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Boundary height correlations in a two-dimensional Abelian sandpile

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Abstract. Boundary height distributions of the two-dimensional Abelian sandpile model are studied in the self-organized critical state. All height probabilities are calculated explicitly both at open and closed boundaries. The leading asymptotic term of the corresponding correlation functions is observed to behave as r^{-4} when $r \to \infty$. On the basis of conformal field theory predictions the bulk height correlators are shown to have the same critical exponents as boundary ones. All heights seem to be identified with appropriate counterparts of the local energy operator in the zero-component Potts model.

1. Introduction and results

The concept of self-organized criticality (SOC) introduced recently by Bak *et al* [1] is expected to be the underlying cause of a variety of critical phenomena involving dissipative, nonlinear transport in open systems, such as earthquake structure, economics, light pulses from quasars, etc [2, 3]. In these phenomena a system evolves stochastically into a certain critical state. It lacks therein any characteristic length as well as timescale and obeys power-law distributions. The critical state is independent of the initial configuration of the system and, unlike ordinary critical phenomena, no fine tuning is necessary to arrive at this state.

The Abelian sandpile model proposed by Bak *et al* turns out to be the simplest one that captures all essential properties of SOC [4–6]. Another reason why a lot of attention has been paid to this model during last years is that the problem admits a purely analytic treatment [7–11]. The number of distinct recurrent configurations of the Abelian sandpile on an arbitrary lattice is equal to the number of spanning trees on this lattice [9]. As to the latter, it can be derived by making use of the Kirchhoff theorem [12, 13]. Providing an effective tool for investigating sandpiles, this theorem reveals their intimate relation to some statistical models, such as Potts, dense polymers, dimer. As the spatial SOC structure is quite similar to that of critical states in statistical mechanics, the program for studying SOC phenomena has to be along the lines of the usual statistical systems. Together with bulk critical exponents, the problem of determining the same surface quantities is an essential point of this program. It is of especial importance in the two-dimensional case when conformal field theory connects surface and bulk properties of the model [14, 15].

The natural formulation of the Abelian sandpile model is given in terms of integer height variables z_i at each site of a finite square lattice $\mathcal{L}$. In a stable configuration the height z_i at any site $i \in \mathcal{L}$ takes values 1, 2, 3 or 4. Particles are added at randomly chosen sites and the addition of a particle increases the height at that site by one. If this height exceeds the critical value Δ_{ii}, then the site topples, and on toppling its height decreases by Δ_{ii} and the heights at each of its neighbours j increases by $-\Delta_{ij}$. Here Δ is a discrete Laplacian matrix specifying the toppling rules with the following elements:

$$\Delta_{ij} = \begin{cases} 4 & i = j \\ -1 & |i-j| = 1 \\ 0 & \text{otherwise}. \end{cases} \tag{1}$$

For proper sites i, j in the bulk of the lattice, the symbol $|i-j|$ denotes the distance between sites i and j.

To formulate the toppling rules at the boundary $\partial\mathcal{L}$ of the sandpile, we can set up two standard boundary-value problems for the Laplacian on a finite lattice $\mathcal{L}$.

(i) *Dirichlet* problem. Open boundary conditions, when $\Delta_{ii} = 4$ for $i \in \partial\mathcal{L}$, and, therefore, the sand particles are allowed to leave the system through the boundary.

(ii) *Neumann* problem. Closed boundary conditions, when $\Delta_{ii} = 3$ for $i \in \partial\mathcal{L}$, and the sand particles cannot leave the system through $\partial\mathcal{L}$.

If all the boundaries are closed, any steady state of the sandpile is certainly impossible. Therefore, in considering the half-plane geometry, we start with a rectangular shape of the lattice $\mathcal{L}$. Then, we impose open boundary conditions on all edges of the rectangle except one, where the boundary conditions of the types mentioned above are assumed. Further, we shift to this edge the zeros of column and row numbers and let all sizes of the lattice tend to infinity. In other words, we shall call *boundary* sites of the lattice those which are placed on the 'line edge' far from the corners of the lattice.

The spatial structure of SOC state is completely characterized by the set of all correlation functions: the probabilities $\mathcal{P}_a$, $a \in \{1, 2, 3, 4\}$ of finding the value $z_i = a$ at a given lattice site i, two-point correlators $\mathcal{P}_{ab}(r)$ for any sites $i, j \in \mathcal{L}$ of respective heights a and b at a distance r apart, etc. The bulk height correlation function for unit height was studied by Majumdar and Dhar [8]. To calculate all other bulk height probabilities, Priezzhev [10] has recently developed a complicated technique based on θ-graph enumeration. He has observed that, in spite of the local nature of height variables of a sandpile, their tree representations are essentially non-local with the exception of unit height. Unfortunately, θ-graphs are too complex to be calculated explicitly, even for two-point correlation functions. That is the reason why bulk height correlators have yet to be derived. Conformal field theory, nevertheless, predicts an intimate relation between boundary and bulk properties of the model. So we may hope to obtain some information about the bulk correlators from the boundary ones.

Studying the boundary effects in the Abelian sandpile has been initiated by our previous work [11]. However, only the unit height has been investigated therein. The purpose of this paper is to find all Boundary height correlation functions. We observe that, though Boundary height variables cannot be directly represented as local tree diagrams, they may be calculated as their linear combinations without any θ-graphs. Our results are as follows.

(i) The probabilities $\mathcal{P}_a$ are obtained:
at an *open* boundary

$$\mathcal{P}_1 = \frac{9}{2} - \frac{42}{\pi} + \frac{320}{3\pi^2} - \frac{512}{9\pi^3} \approx 0.103\,823 \tag{2}$$

$$\mathcal{P}_2 = -\frac{33}{4} + \frac{66}{\pi} - \frac{160}{\pi^2} + \frac{1024}{9\pi^3} \approx 0.216\,571 \tag{3}$$

$$\mathcal{P}_3 = \frac{15}{4} - \frac{22}{\pi} + \frac{160}{3\pi^2} - \frac{512}{9\pi^3} \approx 0.316\,225 \tag{4}$$

$$\mathcal{P}_4 = 1 - \frac{2}{\pi} \approx 0.363\,380 \tag{5}$$

at a *closed* boundary

$$\mathcal{P}_1 = \frac{3}{4} - \frac{2}{\pi} \approx 0.113\,380 \tag{6}$$

$$\mathcal{P}_2 = \frac{1}{\pi} \approx 0.318\,309 \tag{7}$$

$$\mathcal{P}_3 = \frac{1}{4} + \frac{1}{\pi} \approx 0.568\,309\,. \tag{8}$$

(ii) The two-point correlation functions $\mathcal{P}_{ab}(r)$ decay according to the law: at an *open* boundary

$$\mathcal{P}_{11}(r) = \mathcal{P}_1^2 + \left(-\frac{9}{\pi^2} + \frac{160}{\pi^3} - \frac{9472}{9\pi^4} + \frac{81\,920}{27\pi^5} - \frac{262\,144}{81\pi^6}\right)\frac{1}{r^4} + \cdots \tag{9}$$

$$\mathcal{P}_{12}(r) = \mathcal{P}_1\mathcal{P}_2 + \left(\frac{27}{\pi^2} - \frac{440}{\pi^3} + \frac{23\,680}{9\pi^4} - \frac{20\,480}{3\pi^5} + \frac{524\,288}{81\pi^6}\right)\frac{1}{r^4} + \cdots \tag{10}$$

$$\mathcal{P}_{13}(r) = \mathcal{P}_1\mathcal{P}_3 + \left(-\frac{21}{\pi^2} + \frac{920}{3\pi^3} - \frac{14\,720}{9\pi^4} + \frac{102\,400}{27\pi^5} - \frac{262\,144}{81\pi^6}\right)\frac{1}{r^4} + \cdots \tag{11}$$

$$\mathcal{P}_{14}(r) = \mathcal{P}_1\mathcal{P}_4 + \left(\frac{3}{\pi^2} - \frac{80}{3\pi^3} + \frac{512}{9\pi^4}\right)\frac{1}{r^4} + \cdots \tag{12}$$

$$\mathcal{P}_{22}(r) = \mathcal{P}_2^2 + \left(-\frac{81}{\pi^2} + \frac{1200}{\pi^3} - \frac{58\,432}{9\pi^4} + \frac{409\,600}{27\pi^5} - \frac{1\,048\,576}{81\pi^6}\right)\frac{1}{r^4} + \cdots \tag{13}$$

$$\mathcal{P}_{23}(r) = \mathcal{P}_2\mathcal{P}_3 + \left(\frac{63}{\pi^2} - \frac{2480}{3\pi^3} + \frac{35\,776}{9\pi^4} - \frac{225\,280}{27\pi^5} + \frac{524\,288}{81\pi^6}\right)\frac{1}{r^4} + \cdots \tag{14}$$

$$\mathcal{P}_{24}(r) = \mathcal{P}_2\mathcal{P}_4 + \left(-\frac{9}{\pi^2} + \frac{200}{3\pi^3} - \frac{1024}{9\pi^4}\right)\frac{1}{r^4} + \cdots \tag{15}$$

$$\mathcal{P}_{33}(r) = \mathcal{P}_3^2 + \left(-\frac{49}{\pi^2} + \frac{560}{\pi^3} - \frac{21\,568}{9\pi^4} + \frac{40\,960}{9\pi^5} - \frac{262\,144}{81\pi^6}\right)\frac{1}{r^4} + \cdots \tag{16}$$

$$\mathcal{P}_{34}(r) = \mathcal{P}_3\mathcal{P}_4 + \left(\frac{7}{\pi^2} - \frac{40}{\pi^3} + \frac{512}{9\pi^4}\right)\frac{1}{r^4} + \cdots \tag{17}$$

$$\mathcal{P}_{44}(r) = \mathcal{P}_4^2 + \left(-\frac{1}{\pi^2}\right)\frac{1}{r^4} + \cdots \tag{18}$$

at a *closed* boundary

$$\mathcal{P}_{11}(r) = \mathcal{P}_1^2 + \left(-\frac{9}{\pi^2} + \frac{48}{\pi^3} - \frac{64}{\pi^4}\right)\frac{1}{r^4} + \cdots \tag{19}$$

$$\mathcal{P}_{12}(r) = \mathcal{P}_1\mathcal{P}_2 + \left(\frac{12}{\pi^2} - \frac{68}{\pi^3} + \frac{96}{\pi^4}\right)\frac{1}{r^4} + \cdots \tag{20}$$

$$\mathcal{P}_{13}(r) = \mathcal{P}_1\mathcal{P}_3 + \left(-\frac{3}{\pi^2} + \frac{20}{\pi^3} - \frac{32}{\pi^4}\right)\frac{1}{r^4} + \cdots \tag{21}$$

$$\mathcal{P}_{22}(r) = \mathcal{P}_2^2 + \left(-\frac{61}{4\pi^2} + \frac{96}{\pi^3} - \frac{144}{\pi^4}\right)\frac{1}{r^4} + \cdots \tag{22}$$

$$\mathcal{P}_{23}(r) = \mathcal{P}_2\mathcal{P}_3 + \left(\frac{13}{4\pi^2} - \frac{28}{\pi^3} + \frac{48}{\pi^4}\right)\frac{1}{r^4} + \cdots \tag{23}$$

$$\mathcal{P}_{33}(r) = \mathcal{P}_3^2 + \left(-\frac{1}{4\pi^2} + \frac{8}{\pi^3} + \frac{16}{\pi^4}\right)\frac{1}{r^4} + \cdots\,. \tag{24}$$

(iii) Since all Boundary height correlators have the same critical exponent $x_\parallel = 2$, they should correspond to the same quantity in a scaling limit, i.e. all heights can be identified with the same conformal field in an appropriate conformal field theory. Now, for all bulk height correlators to be described, it suffices to study any one of them in detail. The two-point bulk-correlation function $\mathcal{P}_{11}(r)$ for unit heights calculated by Majumdar and Dhar [8] decays as r^{-4} for large r. Thus, we may conclude that the only critical exponent which corresponds, as mentioned above, to all bulk height correlators equals $x = 2$. Majumdar and Dhar [9] obtained a correspondence between the proper sets of configurations of a sandpile and the zero-component Potts model. In this way, the critical exponent $x = 2$ corresponds to the energy–energy correlator in the latter model.

2. Trees representation of height variables

The principal concept which provides a one-to-one correspondence between sandpile configurations and spanning trees on the lattice $\mathcal{L}$ is that of a forbidden subconfiguration (FSC). In accordance with Dhar [7] we define FSC as a set of lattice sites $\mathcal{F} \in \mathcal{L}$, the corresponding heights of which satisfy the following inequalities: $z_j \leqslant$ number of the nearest neighbours of site j in the subset $\mathcal{F}$. The configurations not containing any FSC are referred to as allowed configurations. Dhar has shown [7] that the number of distinct allowed configurations in the SOC is given by the following simple formula:

$$\mathcal{N}_A = \det \Delta \tag{25}$$

where Δ is the matrix of the toppling rules (i.e. discrete Laplacian).

The following algorithm for finding FSC *in a given configuration* C *has been proposed:* consider a number of subsets $\mathcal{F}_0, \mathcal{F}_1, \mathcal{F}_2, \ldots$, the initial one $\mathcal{F}_0$ being the lattice $\mathcal{L}$ itself. Choose any site from $\mathcal{F}_0$ wherein the height exceeds the number of its nearest neighbours. *Eliminating this site we obtain the next subset* $\mathcal{F}_1$. *The remaining elements of the sequence* $\mathcal{F}_1, \mathcal{F}_2, \ldots$ are constructed in a similar way. If the lattice becomes empty as a result of this procedure, the initial configuration is allowed. Otherwise we obtain a certain subset $\mathcal{F}$, wherefrom any site cannot be thrown out. This subset is nothing but FSC. Dhar called this procedure the 'burning' algorithm. After a slight modification it enables one to draw the spanning tree for a given allowed configuration. Moreover, there exists a one-to-one correspondence between allowed sandpile configurations and spanning trees. One can use this fact to compute the height probabilities.

Following Priezzhev [10] we subdivide, fixing the site $i_0 \in \mathcal{L}$, the set of all allowed configurations into four parts $\mathcal{S}_1, \mathcal{S}_2, \mathcal{S}_3, \mathcal{S}_4$, which are defined as follows: the configuration C belongs to

$\mathcal{S}_1$, iff it remains allowed under any substitution $z_0 = 1, 2, 3, 4$, into i_0;
$\mathcal{S}_2$, iff it remains allowed when $z_0 = 2, 3, 4$, but becomes forbidden when $z_0 = 1$;
$\mathcal{S}_3$, iff it remains allowed when $z_0 = 3, 4$ but becomes forbidden when $z_0 = 1, 2$;
$\mathcal{S}_4$, iff it remains allowed only when $z_0 = 4$.

The fact that for all admitted substitutions the number of configurations is equal leads to the following expressions for height probabilities:

$$\mathcal{P}_1 = \frac{\mathcal{N}_1}{4\mathcal{N}_A} \qquad \mathcal{P}_2 = \mathcal{P}_1 + \frac{\mathcal{N}_2}{3\mathcal{N}_A} \qquad \mathcal{P}_3 = \mathcal{P}_2 + \frac{\mathcal{N}_3}{2\mathcal{N}_A} \qquad \mathcal{P}_4 = \mathcal{P}_3 + \frac{\mathcal{N}_4}{\mathcal{N}_A}. \tag{26}$$

Here $\mathcal{N}_i$ is the number of configurations in the subset $\mathcal{S}_i$. When the site i_0 is located at the open boundary a general description of these subsets remains the same, but at the closed boundary (where critical height equals 3) the set $\mathcal{S}_4$ is empty. Height probabilities in this case are expressed as follows:

$$\mathcal{P}_1 = \frac{\mathcal{N}_1}{3\mathcal{N}_A} \qquad \mathcal{P}_2 = \mathcal{P}_1 + \frac{\mathcal{N}_2}{2\mathcal{N}_A} \qquad \mathcal{P}_3 = \mathcal{P}_2 + \frac{\mathcal{N}_3}{\mathcal{N}_A}. \tag{27}$$

To formulate the Kirchhoff theorem, we now recall some definitions from graph theory.

(a) A connected subgraph of a graph $\mathcal{L}$ which contains all its sites and has no cycles is a *spanning tree*.
(b) A spanning tree with one site (the *root*) distinguished from all other sites by this only fact is called a *rooted* spanning tree.
(c) Since the rooted spanning tree is a connected graph, there is a path from every site of it to the root. We may *orient* this path so that all its bonds will have arrows in the direction to the root. The acyclic property of the spanning tree provides the consistency of this procedure for all bonds of the tree.
(d) If an oriented path from site i to the root passes through site j, then site i is called a *predecessor* of site j. A subtree containing all predecessors of a given site i will be referred to as a branch of the spanning tree growing from the site i. Leaf is a branch of the spanning tree containing only one of its bonds.

Kirchhoff theorem. If to any bond of the graph $\mathcal{L}$, whose adjacent sites i and j are different from the root, the weight x_{ij} is ascribed, then the determinant of the matrix

$$\Delta_{ij}(x) = \begin{cases} \sum_k x_{ik} & i = j \\ -x_{ij} & i \text{ and } j \text{ are adjacent sites} \\ 0 & \text{otherwise} \end{cases} \tag{28}$$

is a generating function of the rooted spanning trees on this graph. In particular, when $x_{ij} = 1$ for all i and j, this matrix coincides with the discrete Laplacian and its determinant gives the total number of rooted spanning trees.

Now we may redefine the subsets $\mathcal{S}_1, \mathcal{S}_2, \mathcal{S}_3, \mathcal{S}_4$ in terms of spanning trees. Let us first describe the subset $\mathcal{S}_1$. Here, for every allowed configuration the burning procedure eliminates the site i_0 only after all its nearest neighbours. Hence the number of sandpile configurations of the subset $\mathcal{S}_1$ equals the number of spanning trees which have just the only leaf attached to i_0 and point in the direction of one of its nearest neighbours.

Consider now the subset $\mathcal{S}_2$. By definition, the substitution $z_0 = 1$ transforms an arbitrary configuration $\mathcal{C}$ of this subset into a forbidden one. The resulting FSC comprises the site i_0 and only one of its nearest neighbours (if it is not the case, then this configuration would be forbidden under the substitution $z_0 = 2$ as well). Let $\mathcal{F}(\mathcal{C})$ be the FSC obtained by the substitution $z_0 = 1$ in $\mathcal{C}$. As above, the burning procedure first eliminates the nearest neighbours of site i_0 which do not belong to $\mathcal{F}(\mathcal{C})$, second the site i_0 itself, and finally the others sites of FSC. Therefore, the FSC is represented as a branch of the spanning tree, growing from the site i_0 and containing only one of its nearest neighbours which is a predecessor of i_0. The others do not belong to the FSC and so they cannot be predecessors of site i_0. The total number of such trees is just equal to N_2.

The description of subsets $\mathcal{S}_3$ (as well as $\mathcal{S}_4$) is quite similar to $\mathcal{S}_2$. The only difference is that in these cases the predecessors of the site i_0 are taken to be two (and three, respectively) nearest neighbours of this site.

3. Local and non-local tree diagrams

The Kirchhoff theorem provides an effective tool to calculate local tree diagrams. Although the method of handling them is far from being novel, it is worth recalling its principal ideas.

Any modification of the weights of a finite number of lattice bonds is called a local defect of the lattice. For example, deleting the bonds or inserting additional ones can be considered as a proper local defect. The difference between a discrete Laplacian of the new lattice Δ' and that of the initial one Δ is referred to as the defect matrix δ. The locality condition implies simply that only a finite number of the rows and columns of the defect matrix δ have non-zero elements.

Another important concept is that of a local tree diagram. We define it as a finite set of black and white arrows on the lattice bonds. Any spanning tree passing through all black arrows but not through the white ones is called compatible with this diagram. Given a local tree diagram, the problem we are interested in is to find the total number of compatible spanning trees. To solve this problem, we have to construct a matrix of the proper local defect by setting the weights of all bonds with white arrows to zero; whereas those with black arrows, to ϵ. Then, according to the Kirchhoff theorem, the highest term of the polynomial $\det \Delta'(\epsilon)$ is just the number we are interested in. So the ratio of the number of spanning trees compatible with a given diagram and the number of all trees on the lattice $\mathcal{L}$ is given by an easily calculable determinant

$$\mathrm{Prob}(\delta) = \lim_{\epsilon\to\infty} \frac{\det \Delta'}{\epsilon^n \det \Delta} = \lim_{\epsilon\to\infty} \det(\mathbf{1} + G\delta)/\epsilon^n \tag{29}$$

where n is the number of black arrows in the diagram, $\mathbf{1}$ is the unit matrix and $G = \Delta^{-1}$ is the lattice Green function which depends on the boundary value problem for this Laplacian.

For two such problems mentioned in the introduction the corresponding Green functions are given by the following expressions [11]:
for *open* boundary conditions

$$G_{\mathrm{open}}(n_1, n_2; m) = \frac{1}{\pi^2} \int_0^\pi\!\!\int_0^\pi \frac{\sin n_1\beta \sin n_2\beta \cos m\alpha}{2 - \cos\beta - \cos\alpha}\, \mathrm{d}\alpha\, \mathrm{d}\beta \tag{30}$$

for *closed* boundary conditions

$$G_{\mathrm{closed}}(n_1, n_2; m) = \frac{1}{\pi^2} \int_0^\pi\!\!\int_0^\pi \frac{\cos(n_1 - 1/2)\beta \cos(n_2 - 1/2)\beta \cos m\alpha - 1}{2 - \cos\beta - \cos\alpha}\, \mathrm{d}\alpha\, \mathrm{d}\beta\,. \tag{31}$$

Here $n_1 = 1, 2, \ldots$ labels the column and $m_1 = \ldots -2, -1, 0, 1, 2, \ldots$ labels the row of site $i_1 = (n_1, m_1) \in \mathcal{L}$ and $n_2 = 1, 2, \ldots$ labels the column and $m_2 = \ldots -2, -1, 0, 1, 2, \ldots$ labels the row of site $i_2 = (n_1, m_1) \in \mathcal{L}$; m denotes the distance between sites i_1 and i_2 along the boundary, $m = m_1 - m_2$.

Unfortunately, one cannot construct any reasonable local tree diagrams to represent height variables directly (except unit height). The problem is that height probabilities are determined by the number of predecessors amongst nearest neighbours of a given site. Local tree diagrams do not contain information like that.

For our purpose we introduce here the concept of non-local tree diagrams. First we choose any site $i_0 \in \mathcal{L}$ to be considered as a central site of the diagram. Now we define a non-local diagram as a finite set of black and white arrows on the lattice bonds (as for a local tree diagram) and, in addition, as a collection of black and white circles on the lattice sites. If a spanning tree passes through all black arrows but not through the white ones, and the

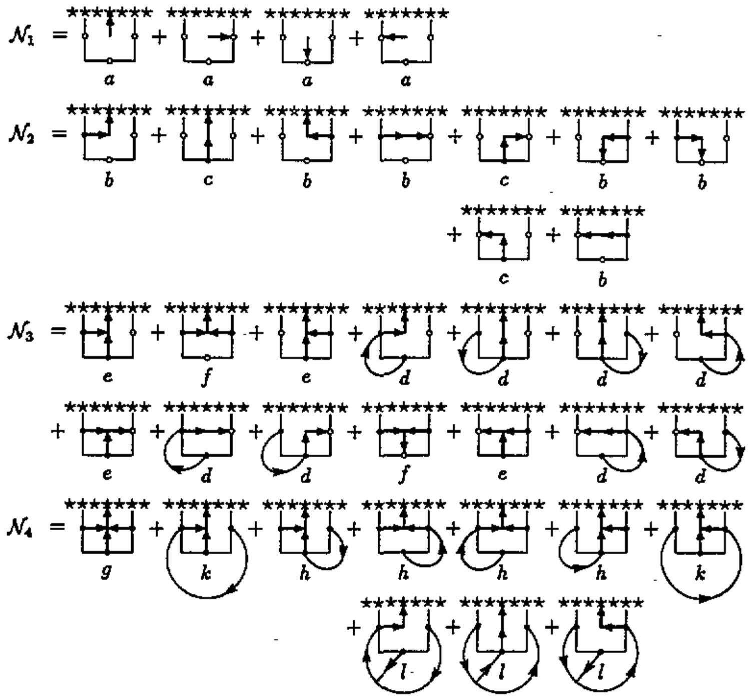

Figure 1. Diagram representation of $\mathcal{S}_1, \mathcal{S}_2, \mathcal{S}_3, \mathcal{S}_4$ at the open boundary of the lattice.

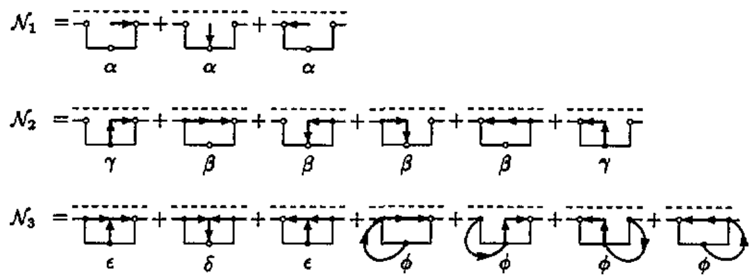

Figure 2. Diagram representation of $\mathcal{S}_1, \mathcal{S}_2, \mathcal{S}_3$ at the closed boundary of the lattice.

sites with black circles are predecessors of the central site, but those with white circles are not, then such a spanning tree will be called compatible with a given non-local tree diagram.

The tree representations of height probabilities at the open and closed boundaries of the lattice are presented in figures 1 and 2. Here every fragment of the lattice represents, symbolically, the number of spanning trees compatible with a given non-local tree diagram. Stars denote an open boundary of the sandpile where sand grains are allowed to leave the system. In terms of spanning trees these sites are nothing but roots of the trees. The broken line in figure 2 denotes a closed boundary of the sandpile. Full lines are the bonds of the lattice. We eliminate the bonds having two opposite white arrows on them. Black arrows are oriented tree bonds, as mentioned above. Full circles are predecessors of the central site i_0 but open ones are not.

There is no general method to calculate non-local diagrams. It is surprising indeed that

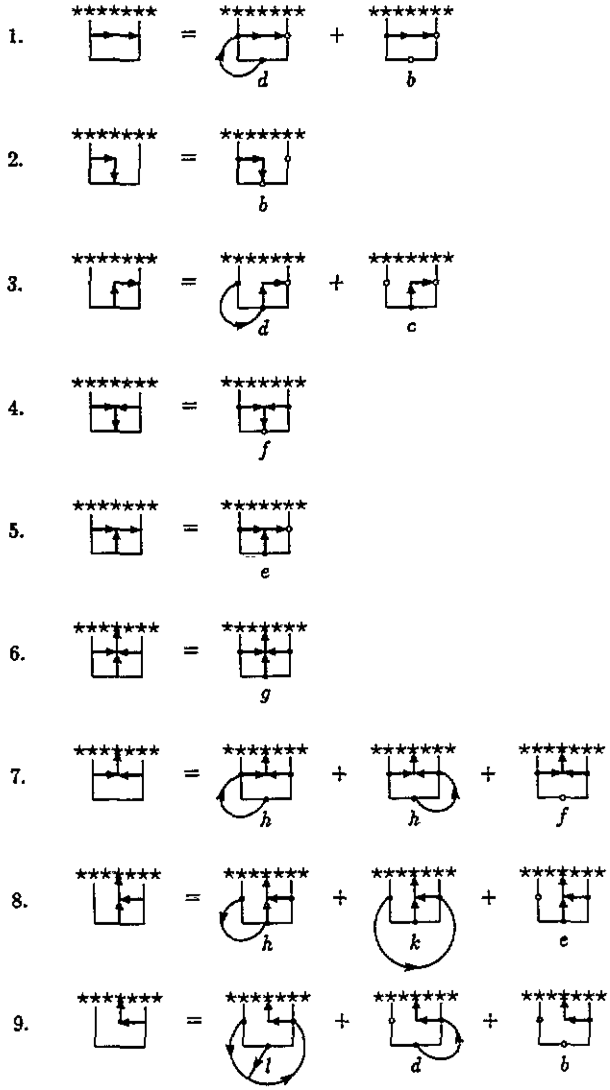

Figure 3. A system of linear equations on the non-local variables at the open boundary of the sandpile.

those of them which appear in the definition of Boundary height probabilities are simply expressed in terms of local tree diagrams. To calculate them, we first decompose local tree diagrams into non-local ones, as is clear from figures 3 and 4. As a result, we obtain a system of linear equations for the non-local variables (numbers of spanning trees compatible with non-local tree diagrams in these figures). Their number, however, is too large for this system to be solved immediately.

For further progress we must find out some relationships between the non-local diagrams themselves. These relationships do exist. Important for us is the following point. Let us consider the non-local diagrams in figure 5. Different though these diagrams are, the numbers of spanning trees compatible with them are equal. Really, we may reverse all

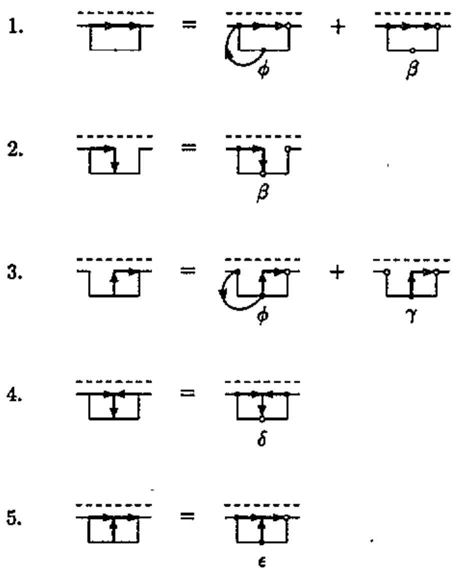

Figure 4. A system of linear equations on the non-local variables at the closed boundary of the sandpile.

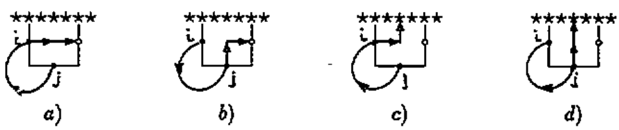

Figure 5. Though these diagrams look different, the number of spanning trees compatible with them are equal.

arrows of a spanning tree along the path joining sites i and j in figure 5(a) and switch at the same time an arrow pointing to i_0 from the position i to the position j, thus obtaining a spanning tree compatible with diagram in figure 5(b). So, we may transform every spanning tree compatible with one of these diagrams into that compatible with another. These arguments are quite general indeed. Now one may reduce a variety of non-local diagrams in figures 3 and 4 to a small number of distinct non-local variables labelled by latin and greek letters for open and closed boundary conditions, respectively.

Finally, the following systems of linear equations are obtained for the non-local variables:
for *open* boundary conditions

$$\mathrm{Prob}(1) = \frac{512}{9\pi^3} - \frac{352}{3\pi^2} + \frac{60}{\pi} - 9 = d + b \tag{32}$$

$$\mathrm{Prob}(2) = \frac{512}{9\pi^3} - \frac{256}{3\pi^2} + \frac{38}{\pi} - \frac{21}{4} = b \tag{33}$$

$$\mathrm{Prob}(3) = \frac{512}{9\pi^3} - \frac{128}{\pi^2} + \frac{54}{\pi} - 6 = d + c \tag{34}$$

$$\mathrm{Prob}(4) = -\frac{512}{9\pi^3} + \frac{64}{\pi^2} - \frac{24}{\pi} + 3 = f \tag{35}$$

$$\mathrm{Prob}(5) = -\frac{512}{9\pi^3} + \frac{416}{3\pi^2} - \frac{76}{\pi} + 12 = e \tag{36}$$

$$\mathrm{Prob}(6) = \frac{2048}{9\pi^3} - \frac{192}{\pi^2} + \frac{152}{3\pi} - 4 = g \tag{37}$$

$$\mathrm{Prob}(7) = -\frac{512}{3\pi^3} - \frac{1280}{9\pi^2} - \frac{112}{3\pi} + 3 = 2h + f \tag{38}$$

$$\mathrm{Prob}(8) = -\frac{512}{3\pi^3} + \frac{1472}{9\pi^2} - \frac{170}{3\pi} + 7 = h + k + g \tag{39}$$

$$\mathrm{Prob}(9) = \frac{1024}{9\pi^3} - \frac{1024}{9\pi^2} + \frac{124}{3\pi} - \frac{21}{4} = b + d + l \tag{40}$$

and for *closed* boundary conditions

$$\mathrm{Prob}(1) = -\frac{2}{\pi} + \frac{3}{4} = \beta + \phi \tag{41}$$

$$\mathrm{Prob}(2) = -\frac{1}{2\pi} + \frac{1}{4} = \beta \tag{42}$$

$$\mathrm{Prob}(3) = \frac{5}{2\pi} - \frac{3}{4} = \gamma + \phi \tag{43}$$

$$\mathrm{Prob}(4) = \frac{1}{\pi} - \frac{1}{4} = \delta \tag{44}$$

$$\mathrm{Prob}(5) = \frac{5}{2\pi} - \frac{3}{4} = \epsilon. \tag{45}$$

Here $\mathrm{Prob}(k)$ is the ratio of the number of spanning trees compatible with a local tree diagram, labelled by the number k in figures 3 and 4 to the total number of spanning trees on the lattice $\mathcal{L}$. Calculation of the Green functions (30), (31) and determinants (29) for each of these local tree diagrams leads to the above presented equations.

A direct solution of the last two systems gives us the expressions (2)–(8) for the height probabilities, presented in the introduction.

4. Height correlations

Calculation of the two-point correlation functions is quite similar in principle but more tedious in practice. Here we underline the principal distinctions from the previous case. Fixing the sites $i_0, j_0 \in \partial\mathcal{L}$, we must subdivide the set of all allowed configurations into components $\mathcal{S}_{kl}$ which are defined as follows. The configuration C belongs to $\mathcal{S}_{kl}$, iff it remains allowed when $z_{i_0} \geqslant k$ and $z_{j_0} \geqslant l$, but becomes forbidden otherwise. Here $k, l \in \{1, 2, 3, 4\}$ for open boundary conditions and $k, l \in \{1, 2, 3\}$ for closed ones. Since the number of configurations, obtained by all admitted substitutions, is equal, we result in the following expressions for the correlation functions:
at an *open* boundary

$$\mathcal{P}_{kl} = \mathcal{P}_{k-1l} + \mathcal{P}_{kl-1} - \mathcal{P}_{k-1l-1} + \frac{\mathcal{N}_{kl}/\mathcal{N}_A}{(5-k)(5-l)} \tag{46}$$

at a *closed* boundary

$$\mathcal{P}_{kl} = \mathcal{P}_{k-1l} + \mathcal{P}_{kl-1} - \mathcal{P}_{k-1l-1} + \frac{\mathcal{N}_{kl}/\mathcal{N}_A}{(4-k)(4-l)}. \tag{47}$$

Here $\mathcal{N}_{kl}$ is the number of configurations in the subset $\mathcal{S}_{kl}$.

The spanning-tree representation of the subsets S_{kl} is just the same as in the case of height probabilities. The substitution $z_{i_0} = k - 1$ or $z_{j_0} = l - 1$ converts an arbitrary configuration C from the subset S_{kl} into a forbidden one. The appearing FSC has either a structure with two branches of the spanning tree growing from the sites i_0 and j_0 or a structure with one branch covering both of these sites.

Placing any pair of local tree diagrams which appear in our previous consideration of height probabilities at a distance r, we construct a local tree diagram for two-point correlation functions. In complete analogy with the case of height probabilities we may decompose these local tree diagrams into the non-local ones. The resulting linear system of equations is too large to be presented here. It is clear that the number of equations for two-point correlators is approximately the square of the number for height probabilities. Making use of MATHEMATICA [16] we may solve this system for any finite distance r. Then, using asymptotic behaviour of the Green function (see [11]):
for *open* boundary conditions

$$G_{\mathrm{open}}(1,1;r) = \frac{1}{\pi r^2} - \frac{1}{2\pi r^4} + \cdots \tag{48}$$

and for *closed* ones

$$G_{\mathrm{closed}}(1,1;r) = -\frac{1}{\pi}\ln r - \frac{\gamma}{\pi} - \frac{3\ln 2}{2\pi} - \frac{1}{6\pi r^2} - \frac{17}{240\pi r^4} + \cdots \tag{49}$$

we can obtain the leading asymptotic term for two-point correlators presented in the introduction in formulae (9)–(24).

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